QFT II - Problem Set 7

(42) The Higgs and the top

The weak interaction distinguishes between left and right handed particles. As a reminder, the left handed and right handed components of some Dirac spinor Ψ are given by $\Psi_L = \frac{1}{2}(1+\gamma^5)\Psi$ and $\Psi_R = \frac{1}{2}(1-\gamma^5)\Psi$ and $\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^\dagger = \gamma^5$ and $(\gamma^5)^2 = 1$ in our conventions. From the point of view of weak interactions, left handed electron neutrinos $\nu_L^e(x) = \frac{1}{2}(1+\gamma^5)\nu^e(x)$ and electrons $e_L(x) = \frac{1}{2}(1+\gamma^5)e(x)$ are degenerate and form an $SU(2)$ duplet

$$\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix} \rightarrow U(x) \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}$$

whereas the right handed electron is a singlet. The same is true for top and bottom quarks. They also form a left handed duplet

$$\Psi_L = \begin{pmatrix} t_L \\ b_L \end{pmatrix},$$

and $\Psi_L \rightarrow U(x)\Psi_L$. Now, all terms within the action must be invariant under the symmetries. In particular, the mass term must be invariant.

a) Show that

$$m\bar{\Psi}_L\Psi_L = m(\bar{t}_L, \bar{b}_L) \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

is invariant under $SU(2)$.

b) Show that the above mass term nevertheless is useless, because it vanishes identically! *Hint: Remember that $\bar{\chi} = \chi^\dagger\gamma^0$.*

c) A better mass term is provided by the Higgs field with the potential

$$V(x) = \mu^2\Phi^\dagger\Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2,$$

where $\Phi = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$ is composed of two complex fields ϕ_1 and ϕ_2.

Show that the Higgs potential is invariant under $SU(2)$ transformations $\boldsymbol{U}$ if Φ is a $SU(2)$ duplett, i.e. $\Phi(x) \rightarrow \boldsymbol{U}(x)\Phi(x)$

d) Suppose that $\mu^2 < 0$. What is the classical minimum in terms of $\Phi^\dagger\Phi$ in this case?

e) The Euclidian action for a fermion ψ with mass m is

$$S_E = \int_p \bar{\psi}(-\not{p} + im)\psi,$$

but as we have seen a simple mass term won't work for the top and bottom mass. Using the Higgs duplet, construct a mass term including Ψ_L and Ψ_R that is invariant under $SU(2)$ and does not vanish indentically. Add it's hermitian conjugate of the mass term to make the action real.

f) Suppose that the action for top, bottom and Higgs is

$$S_E = \int_p -\bar{\Psi}_L \not{p} \Psi_L - \bar{t}_R \not{p} t_R - \bar{b}_R \not{p} b_R + ih\bar{\Psi}_L \Phi \Psi_R + ih\bar{\Psi}_R \Phi^\dagger \Psi_L + \mu^2 \Phi^\dagger \Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2 \quad (1)$$

$$= \int_p -\bar{t}\not{p} t - \bar{b}\not{p} b + ih\bar{\Psi}_L \Phi \Psi_R + ih\bar{\Psi}_R \Phi^\dagger \Psi_L + \mu^2 \Phi^\dagger \Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2 \quad (2)$$

and let's define the projectors $P_{L,R} = \frac{1}{2}(1 \pm \gamma^5)$ for brevity, e.g. $t_L = P_L t$

i) Suppose that the Higgs field has acquired a constant vacuum expectation value. We choose it cleverly such that $\phi_2(x)$ vanishes. Using this, simplify the action.

ii) Use the relations $t_L(x) = P_L\, t(x)$ etc. to write the action entirely as a function of t, $\bar{t}$ and ϕ_1, ϕ_1^*. You can neglect the kinetic term for the bottom, because it doesn't couple to anything interesting.

iii) We would like to compute top-quark-loop contributions to the Higgs potential. As you know from exercise (39), the formula for Fermions is

$$U = V - \mathrm{Tr} \ln S^{(2)}.$$

Obtain

$$S^{(2)} = \frac{\overrightarrow{\delta}}{\delta \bar{t}} S \frac{\overleftarrow{\delta}}{\delta t}$$

for the action that you have found so far, namely

$$S = \int_p -\bar{t}\not{p} t + ih\left[\phi_1 \bar{t} P_R t + \phi_1^* \bar{t} P_L t\right] + \mu^2 \phi^* \phi + \frac{\lambda}{2}(\phi^*\phi)^2$$

and plug this into the above formula for U.

iv) You could now perform the integral etc., but there is an easier way to proceed. We know that the mass term for Φ is the linear piece when we derive V,

$$\frac{\mathrm{d}V}{\mathrm{d}\phi_1^*} = \mu^2 \phi + \lambda \phi_1^* \phi_1^2.$$

In order to find the mass correction, it thus suffices to derive U with respect to ϕ_1^*. Do this.

v) Let's pick ϕ_1 real, i.e. $\phi_1 = \phi_1^*$. That simplifies the expression you have so far. *Hint:* $P_L + P_R = 1$

vi) Just like for the ordinary dirac propagator, make a modification like $\frac{1}{-\not{p}+im} \to \frac{-\not{p}-im}{p^2+m^2}$.

vii) Perform the trace in the numerator. *Hint:* $Tr(\gamma^\mu) = 0,\ Tr(\gamma^\mu\gamma^5) = 0$.

viii) Without evaluating the remaining $\int_p$ -integration, what is the correction to the Higgs potential from top-quark loops ? In contrast to scalar loops, the correction is negative, i.e. the fermions contribute towards a symmetry breaking of the Higgs field.

(43) Legendre Transforms and Γ

The Legendre Transform $g(p)$ of some function $f(x)$ is defined as the maximum separation between the function $y = f(x)$ and the straight line $y = px$, i.e. defining the separation

$$F(x,p) = px - f(x),$$

the Legendre transform is the maximum

$$g(p) = \max_x F(p,x)$$

a) A function $g(p)$ is convex if for any two points p_1 and p_2 and $\lambda \in [0,1]$

$$g\left(\lambda p_1 + [1-\lambda]p_2\right) \leq \lambda g(p_1) + [1-\lambda] g(p_2)$$

holds. Show that the Legendre transform $g(p)$ is convex.

b) In the case that $f(x)$ is itself convex and differentiable, there is one single maximum of $F(x,p)$, i.e. $\partial F/\partial x = 0$. From this obtain a connection between p and f. Please note that this condition can serve as an implicit definition of $x(p)$.

c) In the case that $f(x)$ is convex and differentiable, we know from (b) that the maximum is attained at $x(p)$. Use this information to simplify the defining relation $g(p) = \max_x F(p,x)$.

d) Suppose that you have a function $f(x_1, x_2, \ldots)$ with the differential

$$\mathrm{d}f = p_1 \mathrm{d}x_1 + p_2 \mathrm{d}x_2 + \ldots$$

where $p_i = \partial f/\partial x_i$ and that the derivative of f with respect to x_1 is p_1, i.e.

$$\frac{\partial f}{\partial x_1} = p_1.$$

What is the total differential of

$$g \equiv p_1 x_1(p_1) - f(x_1(p_1), x_2, \ldots)?$$

In other words, what are the variables that g depends on?

e) In problem (39) and (40) you computed the effective potential in φ^4 theory at one loop level. It turned out to be

$$U(\varphi) = \frac{1}{2}\left[m_0^2 + \frac{3\Lambda^2}{32\pi^2}\lambda_0\right]\varphi^2 + \frac{\lambda}{8}\varphi^4,$$

and depending on the cut-off Λ and the coupling λ_0, $U(\varphi)$ might become a double-well potential for $m_0^2 < 0$.

i) Is this potential a convex function Γ?

ii) Sketch $\Gamma[\varphi]$ at one-loop level and the "true" Γ obtained from the requirement that Γ is a convex function.