QFT II - Problem Set 6

(38) One Loop Effective Potential

Consider the generating functional for some scalar field $\hat{\varphi}$,

$$Z[j(x)] = \int \mathcal{D}\hat{\varphi} \exp\left(-S[\hat{\varphi}] + \int_x j\hat{\varphi}\right).$$

As you know from the lecture, the generating functional for connected Greens functions is

$$W[j(x)] \equiv \ln Z[j(x)],$$

and the expectation value of $\hat{\varphi}$ is given by

$$\varphi(x) = \frac{\delta}{\delta j(x)} W[j(x)].$$

The Legrendre transform with respect to φ is the 1-PI effective action

$$\Gamma[\varphi(x)] \equiv -W[j[\varphi(x)]] + \int_x j(x)\varphi(x).$$

a) Convince yourself that

$$\frac{\delta\Gamma}{\delta\varphi} = j.$$

b) Show that in terms of a shifted fluctuating field $\chi \equiv \hat{\varphi} - \varphi$, the effective action is

$$\Gamma[\varphi] = -\ln \int \mathcal{D}\chi \exp\left(-S[\varphi+\chi] + \int_x \frac{\delta\Gamma}{\delta\varphi}\chi\right) \tag{1}$$

c) Suppose we pick $\varphi(x)$ such that it solves the classical equation of motion, i.e.

$$\frac{\delta S}{\delta\hat{\varphi}}_{|\hat{\varphi}=\varphi} = 0.$$

Expand S around such a saddle point up to second order and use this to rewrite Equation (1) as $\Gamma = S + \ldots$ (please denote the second derivative of S by $S^{(2)}$).

d) Use

$$\int \mathcal{D}\chi \exp\left(-\frac{1}{2}S^{(2)}\chi^2\right) = \frac{const.}{\sqrt{\det S^{(2)}}},$$

and $\ln\det A = \operatorname{tr}\ln A$ to simplify your result.

e) What changes for a complex scalar or fermionic field ?

(39) Mass corrections

a) We would like to compute corrections to the mass. We have defined what we call "mass" by the second derivative of the potential with respect to the field evaluated at the vacuum value of the field at vanishing momentum, i.e. constant field value. The action is

$$S = \int_x \frac{1}{2}\partial_\mu\hat{\varphi}(x)\partial^\mu\hat{\varphi}(x) + \frac{1}{2}m_0^2\hat{\varphi}(x)^2 + \frac{\lambda_0}{8}\hat{\varphi}(x)^4,$$

what is

$$\frac{\delta^2 S}{\delta\hat{\varphi}(p)\delta\hat{\varphi}(p')}_{|\hat{\varphi}=\varphi},$$

i.e what is the inverse (classical) propagator?

b) In order to find an expression for the mass, we can use a constant background field $\varphi(x) = \varphi = const$, for which $\varphi(p) = \varphi\delta(p)$. In doing so, we will only be able to pick up corrections that are independent of momentum, but that's ok, because the mass is defined at vanishing momentum. Write your result in terms of the inverse propagator $G_0^{-1}(q)$ as

$$S^{(2)}(q, q') = G_0^{-1}(q)\delta(q + q')$$

and use the formula from (38) and the lecture

$$\Gamma_{1l} = \frac{1}{2} V_4 \int_q \ln\left(G_0^{-1}\right)$$

to write down the full effective action at one loop, i.e. $\Gamma = S + \Gamma_{1l}$.

c) Express your result in terms of the effective Potential, i.e. bring it into the form

$$\Gamma = \int_x \frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) + U(\varphi)$$

You should find

$$U(\rho) = m_0^2\rho + \frac{\lambda_0}{2}\rho^2 + \frac{1}{2}\int_q \ln\left(q^2 + m_0^2 + 3\lambda_0\rho\right),$$

where $\rho \equiv \frac{1}{2}\varphi^2$.

d) Evaluating at constant background field in the symmetric phase, (but still using the kinetic term, of course), the inverse propagator will be

$$G^{-1}(p) = p^2 + m^2 = p^2 + \frac{\partial U(\rho)}{\partial \rho}_{|\rho=0}.$$

i) Compute m^2 at one loop level in the symmetric phase for which $\varphi = 0$, i.e.

$$m^2 = \frac{\partial U(\varphi)}{\partial \rho}_{|\rho=0}$$

ii) Compute λ at one loop level in the symmetric phase, i.e.

$$\lambda = \frac{\partial^2 U(\varphi)}{\partial \rho^2}_{|\rho=0}$$

and draw a diagram corresponding to the two contributions you'll get.

iii) In the symmetric phase, express $G(p)$ in terms of $G_0(p) = \left[p^2 + m_0^2\right]^{-1}$. *Hint: use* $(1 + x)^{-1} = 1 - x$, *where* x *contains the loop term.*

(40) Symmetry breaking

In (39), you computed the effective potential at one loop order

$$U(\rho) = m_0^2\rho + \frac{\lambda_0}{2}\rho^2 + \frac{1}{2}\int \frac{2\pi^2 q^3 dq}{(2\pi)^4} \ln\left(q^2 + m_0^2 + 3\lambda_0\rho\right)$$

The integral is of the form

$$\int \frac{q^3 dq}{2(2\pi)^2} \ln\left(q^2 + \mu^2\right)$$

a) Perform this integral using an ultra violett cut-off Λ and expand in powers of μ/Λ.

b) The largest term is a constant and won't influence the dynamics. Hence, we discard it here. But what about gravity, shouldn't it see this vacuum energy? Think a bit about this.

c) The leading order term proportional to μ^2 is the second term below:

$$\int \frac{q^3 dq}{2(2\pi)^2} \ln\left(q^2 + \mu^2\right) = const + \frac{1}{16\pi^2}\Lambda^2\mu^2 + \ldots.$$

Substitute $\mu^2 = m_0^2 + 3\lambda_0\rho$ to obtain an expression for $U(\rho)$.

d) Suppose that $m_0^2 < 0$, how does $U(\varphi)$ look like for small Λ and large Λ ?