QFT II - Problem Set 3

(32) Scattering at work

We would like to gain some more insight into scattering. Let us scatter two complex scalar fields ϕ and χ. The Lagrangian is

$$-S = \int_p (p^2+m^2)\phi^*(p)\phi(p) + (p^2+M^2)\chi^*(p)\chi(p) + \lambda \int_{p_1\ldots p_4} \delta(p_1-p_2+p_3-p_4)\phi(p_1)\phi^*(p_2)\chi(p_3)\chi^*(p_4),$$

where $p^2 = p_\mu p^\mu$ as usual. In terms of annihilation and creation operators, the fields are given by

$$\phi_+(\omega,\boldsymbol{p}) = \frac{1}{\sqrt{2E(\boldsymbol{p})}} a_\phi(\boldsymbol{p}) \quad (1)$$

$$\phi_-(\omega,\boldsymbol{p}) = \frac{1}{\sqrt{2E(\boldsymbol{p})}} b_\phi^\dagger(-\boldsymbol{p}) \quad (2)$$

$$\chi_+(\omega,\boldsymbol{p}) = \frac{1}{\sqrt{2E(\boldsymbol{p})}} a_\chi(\boldsymbol{p}) \quad (3)$$

$$\chi_-(\omega,\boldsymbol{p}) = \frac{1}{\sqrt{2E(\boldsymbol{p})}} b_\chi^\dagger(-\boldsymbol{p}) \quad (4)$$

$$\phi(\omega,\boldsymbol{p}) = \phi_+\,\theta(\omega) + \phi_-\,\theta(-\omega) = \frac{1}{\sqrt{2E(\boldsymbol{p})}}\left[a_\phi(\boldsymbol{p})\,\theta(\omega) + b_\phi^\dagger(\boldsymbol{p})\,\theta(-\omega)\right] \quad (5)$$

$$\chi(\omega,\boldsymbol{p}) = \chi_+\,\theta(\omega) + \chi_-\,\theta(-\omega) = \frac{1}{\sqrt{2E(\boldsymbol{p})}}\left[a_\chi(\boldsymbol{p})\,\theta(\omega) + b_\chi^\dagger(\boldsymbol{p})\,\theta(-\omega)\right], \quad (6)$$

where $a_\phi^\dagger$ and a_ϕ create and destrory ϕ particles and $b_\phi^\dagger$ and b_ϕ create and destroy ϕ antiparticles. The same is true for the χ field. Please note that the hermitian conjugate operators of the fields are simply

$$\phi^\dagger(\omega,\boldsymbol{p}) = \phi_+^\dagger\,\theta(\omega) + \phi_-^\dagger\,\theta(-\omega) \quad (7)$$

$$\chi^\dagger(\omega,\boldsymbol{p}) = \chi_+^\dagger\,\theta(\omega) + \chi_-^\dagger\,\theta(-\omega) \quad (8)$$

So for instance in order to create one ϕ particle, you have to act with the creation operator on the vacuum:

$$a_\phi^\dagger(\boldsymbol{p})|0\rangle = \sqrt{2E}\phi_+^\dagger(\omega,\boldsymbol{p})|0\rangle$$

Let us now scatter four particles: a ϕ particle (momentum $= p_1$), and a χ antiparticle (momentum $= p_2$) coming in and leaving again (momentum of χ leaving is p_3, momentum of ϕ leaving is p_4).

a) Convince yourself that the S-matrix we have to compute is

$$S_{\beta,\alpha} = \left[2E(\boldsymbol{p_1})2E(\boldsymbol{p_2})2E(\boldsymbol{p_3})2E(\boldsymbol{p_4})\right]^{1/2} \lim_{\substack{t'\to\infty \\ t\to-\infty}} \langle\, \phi_+(t',\boldsymbol{p_4})\ \chi_-^\dagger(t',-\boldsymbol{p_3})\ \chi_-(t,-\boldsymbol{p_2})\ \phi_+^\dagger(t,\boldsymbol{p_1})\,\rangle.$$

b) Fourier transform the times t and t'. You should get

$$S_{\beta,\alpha} = \lim_{\substack{t'\to\infty \\ t\to-\infty}} \prod_{i=1\ldots4} \int_{-\infty}^{\infty} \left(\sqrt{2E_i}\frac{\mathrm{d}\omega_i}{2\pi}\right) e^{-it'(\omega_4-\omega_3)} e^{-it(\omega_2-\omega_1)} \langle\, \phi_+(\omega_4,\boldsymbol{p_4})\ \chi_-^\dagger(\omega_3,-\boldsymbol{p_3})\ \chi_-(\omega_2,-\boldsymbol{p_2})\ \phi_+^\dagger(\omega_1,\boldsymbol{p_1})\,\rangle.$$

c) Express ϕ_+ etc. in terms of the fields ϕ and χ. In other words, use $\chi_-^\dagger = \chi^\dagger\theta(-\omega_3)$ etc.

d) Use the θ functions and convince yourself that you can write $S_{\beta,\alpha}$ as

$$S_{\beta,\alpha} = \lim_{\substack{t'\to\infty \\ t\to-\infty}} \prod_{i=1\ldots4} \int_{0}^{\infty} \left(\sqrt{2E_i}\frac{\mathrm{d}\omega_i}{2\pi}\right) e^{-it'(\omega_4+\omega_3)} e^{it(\omega_2+\omega_1)} \langle\, \phi(\omega_4,\boldsymbol{p_4})\ \chi^\dagger(-\omega_3,-\boldsymbol{p_3})\ \chi(-\omega_2,-\boldsymbol{p_2})\ \phi^\dagger(\omega_1,\boldsymbol{p_1})\,\rangle.$$

e) To proceed, we have to compute the 4-point function

$$\langle\, \phi(\omega_4,\boldsymbol{p_4})\ \chi^\dagger(-\omega_3,-\boldsymbol{p_3})\ \chi(-\omega_2,-\boldsymbol{p_2})\ \phi^\dagger(\omega_1,\boldsymbol{p_1})\,\rangle.$$

We know from last term that these can be found by acting with $i\frac{\delta}{\delta J}$ etc. on the generating functional $Z[J]$. As we have two complex fields, we will need two complex sources, say J for ϕ and η for χ:

$$Z[j^\star, j, \eta^\star, \eta] = \int \mathcal{D}\phi\mathcal{D}\chi \exp\left(iS - i\int_p j^*(p)\phi(p) - j(p)\phi^*(p) - \eta^*(p)\chi(p) - \eta(p)\chi^*\right).$$

In principle, we would have to normalize by $Z[j=0]$, but we skip this here. As you have learned last term, you can express $Z[j]$ as

$$Z[j^\star, j, \eta^\star, \eta] = \exp\left(iS_{int}\left[i\frac{\delta}{\delta j^*}, i\frac{\delta}{\delta j}, i\frac{\delta}{\delta \eta^*}, i\frac{\delta}{\delta \eta}\right]\right)\exp\left\{-\int_p j^*\bar{G}_\phi\, j - \int_p \eta^*\bar{G}_\chi\eta\right\},$$

where $\bar{G}_\phi = \frac{-i}{p^2+m^2}$ and $\bar{G}_\chi = \frac{-i}{p^2+M^2}$.

f) Expand the interaction S_{int} to first order in λ.

g) Compute the non-trivial (i.e. scattering) part to this order using

$$\begin{aligned}&\langle\ \phi(\omega_4, \boldsymbol{p_4})\ \chi^\dagger(-\omega_3, -\boldsymbol{p_3})\ \chi(-\omega_2, -\boldsymbol{p_2})\ \phi^\dagger(\omega_1, \boldsymbol{p_1})\ \rangle \\ &\qquad = \left(i\frac{\delta}{\delta j^*(\omega_4, \boldsymbol{p}_4)}\right)\left(i\frac{\delta}{\delta\eta(-\omega_3, -\boldsymbol{p}_3)}\right)\left(i\frac{\delta}{\delta\eta^*(-\omega_2, -\boldsymbol{p}_2)}\right)\left(i\frac{\delta}{\delta j(\omega_1, \boldsymbol{p}_1)}\right)Z[j^\star, j, \eta^\star, \eta]\end{aligned}$$

h) Substitute this expression back into our formula for $S_{\beta,\alpha}$.

i) Just like in the lecture, perform the integrals over ω using the resiudal theorem. For this, use

$$\frac{-i}{p^2+m^2} \to \frac{i}{\omega^2 - E^2(\boldsymbol{p}) + i\epsilon}.$$

j) Think a bit which graphs would contribute to order λ^2.