QFT I - Problem Set 4

(8) Gaussian Integrals

a) Prove the following identity for multi-dimensional integrals over real variables:

$$I_1 = \int dx_1 \cdots dx_n e^{-\frac{1}{2}x_i A_{ij} x_j + x_i J_i} = \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det A}} e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} \tag{1}$$

where A is a real symmetric positive definite matrix (sum convention).

Hints:

1. Complete the square to get rid of the linear $J_i x_i$ piece. It may help you to find the appropriate variable transformation by considering a new variable $y_i \equiv x_i + q_i$ and demanding that $-\frac{1}{2} y_i A_{ij} y_j = -\frac{1}{2} x_i A_{ij} x_j + x_i J_i$, which gives you an expression for y_i in terms of x, A^{-1}, J.

2. Rotate y to diagonalize A, i.e. define a new variable $z \equiv R^{-1} y$ such that $R^{-1} A R$ is diagonal.

3. Use the one-dimensional result $\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$.

b) For integrals over pairs of conjugate complex variables holds a similar identity. Prove:

$$I_2 = \int \prod_{i=1}^{n} \left(\frac{dx_i^* dx_i}{2\pi i} \right) e^{-x_i^* H_{ij} x_j + J_i^* x_i + x_i^* J_i} = [\det H]^{-1} e^{J_i^* H^{-1}_{ij} J_j} \tag{2}$$

for any positive definite Hermitian matrix (all eigenvalues are real and positive). *Hint: Use* $\int \frac{dx^* dx}{2\pi i} e^{-x^* a x} = \int \frac{du dv}{\pi} e^{-a(u^2+v^2)} = \frac{1}{a}$.

(9) Identities

a) We pick up on problem (8) for this identity: Prove the following representation of the inverse of H:

$$(H)^{-1}_{lk} = \det H \int \prod_{i=1}^{n} \left(\frac{dx_i^* dx_i}{2\pi i} \right) x_k^* x_l \, e^{-x_i^* H_{ij} x_j} . \tag{3}$$

Hint: Consider $\left. \frac{\partial^2 I_2}{\partial J_l^* \partial J_k} \right|_{J=0}$.

b) Prove the incredibly important (no kidding!) identity

$$\log \det A = \operatorname{Tr} \log A \tag{4}$$

for a $(n \times n)$ matrix A.